Transport de l'énergie dans le modèle du LMD

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1 Rappel des équations anciennes du MCG du LMD

La coordonnée verticale du modèle est la pression normalisée par sa valeur à la surface: $\sigma = p/p_s$. On utilise en fait σ aux niveaux inter-couches et $s = \sigma^\kappa$ au milieu des couches. On note X et Y les coordonnées horizontales: X (resp. Y) est une fonction biunivoque de la longitude λ (resp. de la latitude ϕ). Les variables scalaires (hentalpie potentielle $\theta = c_p T/{p_s}^\kappa$, géopotentiel Φ et pression de surface p_s) sont évaluées aux points correspondant à des couples de valeurs entières $(X, Y) = (i, j)$. Les variables dynamiques sont décalées par rapport aux variables scalaires en utilisant une grille C dans la définition de Arakawa [1]: le vent zonal est calculé aux points $(X, Y) = (i + 1/2, j)$ et le vent méridien aux points $(X, Y) = (i, j + 1/2)$. La disposition des variables sur la grille est illustrée sur la Figure 1.

On utilise en fait les composantes covariantes ($\tilde{u}$ et $\tilde{v}$) et contravariantes ($\tilde{\tilde{u}}$ et $\tilde{\tilde{v}}$) du vent définies par

$$\begin{array}{llll} \tilde{u} = c_u u & \text{et} \quad \tilde{\tilde{u}} = u/c_u & \text{avec} & c_u = a\cos\phi\,(d\lambda/dX) \\ \tilde{v} = c_v v & \text{et} \quad \tilde{\tilde{v}} = v/c_v & \text{avec} & c_v = a\,(d\phi/dY) \end{array} \tag{1}$$

où u et v sont les composantes physiques du vecteur vent horizontal. On introduit également:

la pression extensive: $\tilde{p}_s$ (pression au sol multipliée par l'aire de la maille).

les trois composantes du flux de masse:

$$U = \overline{\tilde{p}_s}^X \tilde{\tilde{u}},\ V = \overline{\tilde{p}_s}^Y \tilde{\tilde{v}} \text{ et } W = \tilde{p}_s \dot{\sigma} \text{ avec } \dot{\sigma} = \frac{d\sigma}{dt} \tag{2}$$

le facteur de Coriolis multiplié par l'aire de la maille: $f = 2\Omega \sin\phi c_u c_v$ où Ω est la vitesse de rotation de la planète.

la vorticité potentielle absolue:

$$Z = \frac{\delta_x \tilde{v} - \delta_y \tilde{u} + f}{\overline{\tilde{p}_s}^{X,Y}} \tag{3}$$

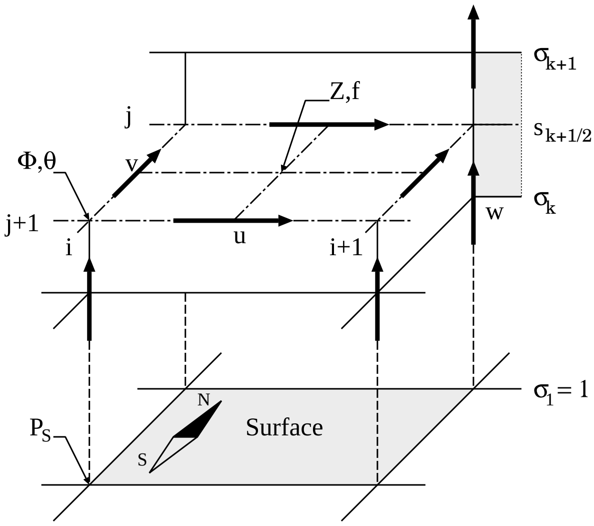

Figure 1: Disposition des variables dans la grille du LMD

l'énergie cinétique

$$K = \frac{1}{2}\left(\overline{\tilde{u}\tilde{\tilde{u}}}^X + \overline{\tilde{v}\tilde{\tilde{v}}}^Y\right) \tag{4}$$

La notation δ_X signifie simplement qu'on effectue la différence entre deux points consécutifs suivant la direction X. La notation $\overline{a}^X$ signifie qu'on prend la moyenne arithmétique de la quantité a suivant la direction X. Les équations discrétisées sont écrites sous la forme suivante:

équations du mouvement:

$$\frac{\partial \tilde{u}}{\partial t} - \overline{Z}^Y \overline{V}^{X,Y} + \delta_x\left(\Phi + K\right) + s\overline{\theta}^X \delta_x {p_s}^\kappa + \frac{\overline{\overline{W}^X \delta_z \tilde{u}}^Z}{\overline{\tilde{p}_s}^X \delta_z \sigma} = S_{\tilde{u}} \tag{5}$$

$$\frac{\partial \tilde{v}}{\partial t} + \overline{Z}^X \overline{U}^{X,Y} + \delta_y\left(\Phi + K\right) + s\overline{\theta}^Y \delta_y {p_s}^\kappa + \frac{\overline{\overline{W}^Y \delta_z \tilde{v}}^Z}{\overline{\tilde{p}_s}^Y \delta_z \sigma} = S_{\tilde{v}} \tag{6}$$

équation thermodynamique:

$$\frac{\partial\left(\tilde{p}_s \theta\right)}{\partial t} + \delta_x\left(\overline{\theta}^X U\right) + \delta_y\left(\overline{\theta}^Y V\right) + \frac{\delta_z\left(\overline{\theta}^Z W\right)}{\delta_z \sigma} = S_\theta \tag{7}$$

équation hydrostatique:

$$\delta_z \Phi = -{p_s}^\kappa \overline{\theta}^z \delta_z s \tag{8}$$

with $\overline{\theta}_1^z = \theta_1$ and $\delta_z s_1 = 1 - s_1$.

équations de continuité:

$$\frac{\partial \tilde{p}_s}{\partial t} + \delta_x U + \delta_y V + \frac{\delta_z W}{\delta_z \sigma} = 0 \tag{9}$$

On a noté S les termes sources dans les différentes équations.

Remarque: on utilise ici la formulation originale, sans contrainte sur la conservation du moment cinétique. Cette formulation est plus proche d'une conservation numérique de l'énergie.

2 Numerical conservation of the total energy

The specific energy is the sum of the specific kinetic energy K, internal energy $I = C_v T$ and potential energy Φ.

2.1 Relationship between potential and internal energy

One important property of the hydrostatic approximation is that the total energy content of a given air column in term of internal energy on one hand and potential energy on the other are proportional

$$\int_0^\infty \Phi \rho dz = \int_0^\infty RT \rho dz \tag{10}$$

This relationship is no more straitforward in the numerical formulation. However the following finite diference relationship

$$\sum_{l=1}^{N} \Phi_l \delta_z \sigma_l - \Phi_s = \sum_{l=1}^{N} RT_l \delta_z \sigma_l = {p_s}^\kappa \sum_{l=1}^{N} \kappa \theta_k s_k \delta_z \sigma_k \tag{11}$$

can be imposed by chosing a particular relationship between the location of the σ and s levels as is shown now.

From the discretized hydrostatic equation Eq. 8, we can write

$$\Phi_l = \Phi_s + {p_s}^\kappa \sum_{k=1}^{l} \left[\overline{\theta}^z\right]_k \delta_z s_k \tag{12}$$

with the convention at the lower boundary that

$$\left[\overline{\theta}^z\right]_1 = \theta_1 \tag{13}$$

The gravitational energy of a complete column of air is thus given by

$$\begin{aligned}
\sum_{l=1}^{N} \Phi_l \delta_z \sigma_l - \Phi_s &= {p_s}^\kappa \sum_{l=1}^{N} \sum_{1 \le k \le l} \left[\overline{\theta}^z\right]_k \delta_z s_k \delta_z \sigma_l && (14)\\
&= {p_s}^\kappa \sum_{k=1}^{N} \sum_{k \le l \le N} \left[\overline{\theta}^z\right]_k \delta_z s_k \delta_z \sigma_l && (15)\\
&= {p_s}^\kappa \sum_{k=1}^{N} \left[\overline{\theta}^z\right]_k \delta_z s_k \sigma_k && (16)\\
&= {p_s}^\kappa \sum_{k=1}^{N} \theta_k \left[\overline{\delta_z s_k \sigma_k}^z\right]_k && (17)
\end{aligned}$$

where

$$\left[\overline{\sigma\delta_z s}^z\right]_1 = \frac{\sigma_2\delta_z s_2}{2} + \delta_z s_1 \tag{18}$$

and

$$\left[\overline{\sigma\delta_z s}^z\right]_N = \frac{\sigma_{N-1}\delta_z s_{N-1}}{2} \tag{19}$$

The Eq. 11 can hence be satisfied by just imposing that $\overline{\sigma\delta_z s}^z = \kappa s\delta_z\sigma$.

2.2 Numerical conservation

We will now derive the time evolution of the total energy content as obtained from the finite diference formulation of the primitive equations. We game is to show that, for adiabatic flows we have

$$\frac{\partial\sum_\sigma \delta_z\sigma \left\langle \tilde{p}_s\left(V^2/2 + C_v T + \Phi\right)\right\rangle}{\partial t} = 0 \tag{20}$$

where $\sum_\sigma$ stands for the summation over all the vertical column of air and $\langle\rangle$ accounts for the summation over all the grid points (the area of the mesh is included in the extensif pressure $\tilde{p}_s$).

Preliminary remark: as usually done in finite difference analysis, the numerical conservation is only concerned with partial and not temporal derivative. The time derivative are suppose to be exact.

The time derivative of the kinetic energy alone can be computed as

$$\frac{\partial K}{\partial t} = \frac{1}{2}\frac{\partial\left[\overline{\tilde{u}\tilde{\tilde{u}}}^x\right] + \left[\overline{\tilde{v}\tilde{\tilde{v}}}^y\right]}{\partial t} \tag{21}$$

$$\frac{1}{2}\frac{\partial\overline{\tilde{u}\tilde{\tilde{u}}}^x}{\partial t} = \overline{\tilde{\tilde{u}}\frac{\partial\tilde{u}}{\partial t}}^x \tag{22}$$

When integrated over the total planet, the classical symmetry property of the averaging operator ($\left\langle a\overline{b}^x\right\rangle = \langle b\overline{a}^x\rangle$) can be applied and one find that

$$\left\langle \tilde{p}_s\frac{\partial K}{\partial t}\right\rangle = \left\langle \overline{\tilde{p}_s}^x\tilde{\tilde{u}}\frac{\partial\tilde{u}}{\partial t}\right\rangle + \left\langle \overline{\tilde{p}_s}^y\tilde{\tilde{v}}\frac{\partial\tilde{v}}{\partial t}\right\rangle \tag{23}$$

$$= \left\langle U\frac{\partial\tilde{u}}{\partial t}\right\rangle + \left\langle V\frac{\partial\tilde{v}}{\partial t}\right\rangle \tag{24}$$

The vorticity term. By choosing a different discretization of the vorticity term [4, 3, 5] the term

$$\langle U\,[ZV]\rangle - \langle V\,[ZU]\rangle \tag{25}$$

would cancel exactly. This is not the case in the present formulation which is based on enstrophy conservation but this term will be forgotten hereafter.

Other terms. The time evolution of the kinetic energy then reduces to

$$\left\langle \tilde{p}_s \frac{\partial K}{\partial t} \right\rangle = - \left\langle U \delta_x (\Phi + K) + sU\overline{\theta}^X \delta_x {p_s}^\kappa + U \frac{\overline{\overline{W}^X \delta_z \tilde{u}}^Z}{\overline{\tilde{p}_s}^X \delta_z \sigma} \right\rangle$$
$$- \left\langle V \delta_y (\Phi + K) + sV\overline{\theta}^Y \delta_y {p_s}^\kappa + V \frac{\overline{\overline{W}^Y \delta_z \tilde{v}}^Z}{\overline{\tilde{p}_s}^Y \delta_z \sigma} \right\rangle \quad (26)$$

By using the antisymmetry property of the partial differential operator $\langle a\delta_x b\rangle = -\langle b\delta_x a\rangle$ (integration by part)

$$\delta_z\sigma \left\langle \tilde{p}_s \frac{\partial K}{\partial t} \right\rangle = \delta_z\sigma \left\langle (\Phi + K)(\delta_x U + \delta_y V) \right\rangle$$
$$+ s\delta_z\sigma \left\langle {p_s}^\kappa \left[\delta_x \left(U\overline{\theta}^x \right) + \delta_y \left(V\overline{\theta}^y \right) \right] \right\rangle \quad (27)$$
$$- \left\langle \overline{\tilde{u}\overline{W}^x \delta_z \tilde{u}}^z \right\rangle - \left\langle \overline{\tilde{v}\overline{W}^y \delta_z \tilde{v}}^z \right\rangle \quad (28)$$

which can be transformed using the continuity and energy equation

$$\delta_z\sigma \left\langle \tilde{p}_s \frac{\partial K}{\partial t} \right\rangle = -\delta_z\sigma \left\langle (\Phi + K) \left(\frac{\partial \tilde{p}_s}{\partial t} + \frac{\partial W}{\partial \sigma} \right) \right\rangle$$
$$- s\delta_z\sigma \left\langle {p_s}^\kappa \left[\frac{\partial (\tilde{p}_s \theta)}{\partial t} + \frac{\delta_z \left(W\overline{\theta}^z \right)}{\delta_z \sigma} \right] \right\rangle$$
$$- \left\langle \overline{\tilde{u}\overline{W}^x \delta_z \tilde{u}}^z \right\rangle - \left\langle \overline{\tilde{v}\overline{W}^y \delta_z \tilde{v}}^z \right\rangle \quad (29)$$

or

$$\sum_\sigma \delta_z\sigma \left\langle \frac{\partial (\tilde{p}_s K)}{\partial t} \right\rangle = -\delta_z\sigma \left\langle \Phi \left[\frac{\partial \tilde{p}_s}{\partial t} + \frac{\delta_z W}{\delta_z \sigma} \right] + s{p_s}^\kappa \left[\frac{\partial (\tilde{p}_s \theta)}{\partial t} + \frac{\delta_z \left(W\overline{\theta}^z \right)}{\delta_z \sigma} \right] \right\rangle$$
$$- \left\langle \overline{\tilde{u}\overline{W}^x \delta_z \tilde{u}}^z + \overline{\tilde{v}\overline{W}^y \delta_z \tilde{v}}^z - K\delta_z W \right\rangle \quad (30)$$

When integrated vertically, the vertical means in the first two terms of the last line disapear and it can then be shown easily that the terms of the last line exactly cancel. This is no more the case indeed when using the new numerical formulation which includes a constraint on the numerical conservation of angular momentum [2]. However, one has to remenmber that the energy is not exactly conserved in both cases because of the discretization of the vorticity term.

The hydrostatic relationship,

$$\sum_\sigma \Phi \delta_z W = -\sum_\sigma W \delta_z \Phi$$
$$= \sum_\sigma W {p_s}^\kappa \overline{\theta}^z \delta_z s \quad (31)$$

leads to the cancellation of two other terms in the first line of Eq. 30:

$$\sum_\sigma \delta_z\sigma \left\langle \Phi \delta_z W + s{p_s}^\kappa \frac{\delta_z \left(W\overline{\theta}^z \right)}{\delta_z \sigma} \right\rangle = 0 \quad (32)$$

Finally, by applying Eq. 11, the evolution of the total kinetic energy can be written as

$$\begin{aligned}\sum_\sigma \delta_z\sigma \left\langle \frac{\partial\left(\tilde{p}_s K\right)}{\partial t}\right\rangle &= -\left\langle \Phi_s \frac{\partial \tilde{p}_s}{\partial t}\right\rangle \\ &\quad -\sum_\sigma \delta_z\sigma \left\langle \kappa {p_s}^\kappa s\theta \frac{\partial \tilde{p}_s}{\partial t} + s{p_s}^\kappa \frac{\partial\left(\tilde{p}_s\theta\right)}{\partial t}\right\rangle\end{aligned}$$

This term is exactly the oposite of the time derivative of the sum of the total internal plus potential energy (or total "total potential" energy) since Eq. 11 insures that

$$\frac{\partial \sum_\sigma \delta_z\sigma\tilde{p}_s\left(C_vT+\Phi\right)}{\partial t} = \Phi_s\frac{\partial p_s}{\partial t} + \sum_\sigma s\delta_z\sigma \frac{\partial\left(\tilde{p}_s {p_s}^\kappa\theta\right)}{\partial t} \tag{33}$$

which leads finally to the numerical conservation of the total energy:

$$\frac{\partial\left[\sum_\sigma \delta_z\sigma\left\langle K+C_vT+\Phi\right\rangle\right]}{\partial t} = 0 \tag{34}$$

Once again, for the particular numerical formulation under consideration, this is only true for non rotational flows.

2.3 Horizontal energy budget

The main part of those derivations are concerned with vertical integrations over a column of air. Therefore, the previous analysis can be used in order to study the horizontal transport of energy by the horizontal circulation. Eq. 24 introduces some numerical error in the energy budget if applied to a given air column. The only other part where horizontal averages or derivatives play a role is for the derivation of Eq. 27 based on the use of the anti-symmetry property of the finite-difference derivative. In fact, this property hides an integration by part:

$$a_1b_1 - a_0b_0 = \frac{\left(a_1-a_0\right)\left(b_1+b_0\right)}{2} + \frac{\left(a_1+a_0\right)\left(b_1-b_0\right)}{2} \tag{35}$$

or

$$\delta_x ab = \overline{a}^X\delta_x b + \delta_x a\overline{b}^X \tag{36}$$

where the term on the left hand side disapears when sumed on the total domain. When sumed on a limited area, the same relation can be applied introducing the boundary terms which are fundamentaly fluxes for our particular application. Therefore, for each time the antisymmetry property is used, a boudary term has to be introudced. The anti-symmerty property is applied for the term of the type $U\delta_x\Phi+K$ and $sU\overline{\theta}^X\delta_x {p_s}^\kappa$ (same thing with V instead of U) so that the time evolution of the total energy of a given air column can be computed as

$$\frac{\partial\left[\sum_\sigma \delta_z\sigma\tilde{p}_s\left(K+C_vT+\Phi\right)\right]}{\partial t} = \sum_\sigma \delta_z\sigma\left[\delta_x\left(U\overline{\Phi+K+s\theta {p_s}^\kappa}^X\right)+\delta_y\left(V\overline{\Phi+K+s\theta {p_s}^\kappa}^Y\right)\right] \tag{37}$$

where the term $U\overline{s\theta {p_s}^\kappa}^X$ accounts for both the advection of the internal energy and the work of the pressure forces on the boundaries.

Warning: this does not give the exact time evolution of the column of air as computed by the model since Eq. 36 is not exactly the one which is used in the derivation of Eq. 27.

References

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